

The Corepresentations of Continuous Groups

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1 Introduction

The theory of corepresentations of non-unitary groups $\mathcal{G} = G + a_0G$, where G denotes a unitary group and a_0 is an antiunitary element, was formulated by Wigner [26], to whom belong the first applications of corepresentations in quantum mechanics. Space groups with antiunitary operations and their corepresentations subsequently found important applications in solid state physics. Lie groups with antilinear operations were not considered.

Wigner's theory of corepresentations was elaborated by a number of authors to the form of a powerful tool for investigations of physical properties of crystals and of magnetic crystals [1, 2, 3, 5, 20, 21]. It was applied in the investigations of symmetry changes at commensurate and incommensurate continuous magnetic phase transitions [8, 9, 10, 11], and to the problem of magnetocrystalline anisotropy of ferromagnetic crystals [14].

It was shown by Birman [1] that a non-unitary symmetry group can intervene in the classical description of a crystal in a state of thermodynamic equilibrium. The non-unitary group of the type $G + KG$, where G is a space group, and where K denotes the operation of complex conjugation, constitutes the complete symmetry group of the crystal-lattice dynamic problem. This group plays the basic role in establishing the one-to-one correspondence between vibration frequencies and irreducible corepresentations (coirreps). The application of the non-unitary group $G + KG$ to a description of lattice vibrations elaborated in [1], opened the way for a further development in this field, made by Kovalev [15, 16, 17, 18, 21] and by Kovalev and Gorbanyuk [20]. These authors formulated another method of demonstrating that there exists the one-to-one correspondence between coirreps and frequencies of

crystal lattice vibrations. As a subsequent step in the exploration of the importance of the non-unitary groups, Kovalev and Gorbanyuk [20], generalized Wigner-Eckart theorem [25] to systems described by magnetic space groups (see [11]).

The corepresentation theory was originally formulated for the case when the subgroup G of the group $G + a_0G$ is unitary [26]. The group $G + a_0G$ then is called a non-unitary group [3], and the element a_0 is an antiunitary operation. The name antiunitary, which was assigned to the antilinear operations of complex conjugation K and of time reversal Θ draws from the fact that when the bilinear product of basis functions is Hermitian, any antiunitary operation is equal to the product of the operation of complex conjugation K with some unitary operation [26]. The name antiunitary does not seem to be appropriate when the operations K or Θ are applied to linear operations which are not unitary, for example to the operations of the group $SL(2, C)$. The modification of group representation theory leading to corepresentations is conditioned by the antilinear character of the operations K or Θ .

In Section 2 we will present the theory of corepresentations without making the assumption that the subgroup G of the group $G + a_0G$ is unitary. Groups of the type $\mathcal{G} = G + a_0G$ will be considered, consisting of the subgroup G which is a group of linear operations and of the coset a_0G , consisting of products of an antilinear operation a_0 with the linear operations belonging to G . The element a_0 itself, in general can be a product of an antilinear operation A with a linear operation g_L^0 , which does not belong to the subgroup G . However, the element g_L^0 has to be of such a type that we have $(Ag_L^0)^2 \in G$. In particular we can have g_L^0 equal to the unit element $\mathbf{1}$.

2 The corepresentation theory of continuous groups

In this Section we are indebted to the presentations of the corepresentation theory for magnetic space groups by Bradley and Cracknell [3], and Kovalev and Gorbanyuk [20].

In the applications of corepresentation theory in quantum mechanics [26], the antiunitary element a_0 was the operation of time-reversal Θ , multiplied by a proper or improper rotation element, represented by a unitary matrix. When the subgroup G need not be unitary, it seems to be misleading to call the group $G + a_0G$ a non-unitary group, and we will not use this name.

Let G be a continuous group of linear transformations which need not be unitary. We define the group

$$\mathcal{G} = G + a_0G \tag{2.1}$$

in which in general the operation a_0 is a product of an antilinear operation with a linear operation, which does not belong to the subgroup G . As the product of any two elements

of the coset a_0G has to belong to G , we must have $a_0^2 \in G$.

Let Γ be an irreducible representation (irrep) of the group G , of dimension d , and let $\varphi_i, i = 1, \dots, d$, be its basis functions. For any element $g \in G$, we then have

$$g\varphi_i = \sum_{j=1}^d \Delta(g)_{ji} \varphi_j \quad \text{or} \quad g\varphi = \tilde{\Delta}(g)\varphi \quad (2.2)$$

where $\Delta(g)$ is the representation matrix, φ is the column matrix constructed from the basis functions $\varphi_1, \dots, \varphi_d$, and $\tilde{\Delta}(g)$ is the transposed matrix. The action of an antilinear operation a_0 on a linear combination of functions φ_i is defined by

$$a_0 \sum_{i=1}^d c_i \varphi_i = \sum_{i=1}^d c_i^* a_0 \varphi_i \quad (2.3)$$

where c_i are complex numbers, and $*$ denotes complex conjugation.

The action of the antilinear element a_0 on the basis functions φ_i leads to another set of functions ϕ_i ,

$$a_0 \varphi_i = \phi_i; \quad i = 1, \dots, d \quad (2.4)$$

We consider this transformation as an endomorphism of the space which is spanned by the functions φ_i . The column matrix constructed from the functions $\phi_i, i = 1, 2, \dots, d$ will be denoted by ϕ . The action of $g \in G$ on ϕ is given by

$$g\phi = ga_0\varphi = a_0(a_0^{-1}ga_0)\varphi = a_0g'\varphi = a_0\tilde{\Delta}(g')\varphi = a_0\tilde{\Delta}(a_0^{-1}ga_0)\varphi = \tilde{\Delta}^*(a_0^{-1}ga_0)\phi \quad (2.5)$$

where the last equality is connected with the antilinear character of a_0 . From Eqs. (2.2) and (2.5) we obtain

$$g \begin{pmatrix} \varphi \\ \phi \end{pmatrix} = \left(\begin{array}{c|c} \tilde{\Delta}(g) & 0 \\ \hline 0 & \tilde{\Delta}^*(a_0^{-1}ga_0) \end{array} \right) \begin{pmatrix} \varphi \\ \phi \end{pmatrix}; \quad \forall g \in G \quad (2.6)$$

We now define the matrix $\overline{\Delta}(g)$ in the representation $\overline{\Gamma}$, by

$$\overline{\Delta}(g) = \Delta^*(a_0^{-1}ga_0); \quad \overline{\Delta}(g) \in \overline{\Gamma} \quad (2.7)$$

where $\overline{\Delta}(g)$ is a matrix representative of $g \in G$, in the representation $\overline{\Gamma}$ of G . This equation defines the representation $\overline{\Gamma}$ of G .

Let a be any element of a_0G , say, a_0g . We then obtain

$$a\varphi = a_0g\varphi = a_0\tilde{\Delta}(g)\varphi = \tilde{\Delta}^*(g)\phi = \tilde{\Delta}^*(a_0^{-1}a)\phi \quad (2.8)$$

where Eqs. (2.2) and (2.4) and the antilinear character of a_0 have been used. We next obtain

$$a\phi = aa_0\varphi = \tilde{\Delta}(aa_0)\varphi \quad (2.9)$$

owing to $aa_0 \in G$. From Eqs. (2.8) and (2.9) we obtain the expression

$$a \begin{pmatrix} \varphi \\ \phi \end{pmatrix} = \left(\begin{array}{c|c} 0 & \tilde{\Delta}^*(a_0^{-1}a) \\ \hline \tilde{\Delta}(aa_0) & 0 \end{array} \right) \begin{pmatrix} \varphi \\ \phi \end{pmatrix} \quad (2.10)$$

If $a = ga_0$, and $g = aa_0^{-1}$, the same formula is obtained, since we have

$$a\varphi = (ga_0)\varphi = \tilde{\Delta}^*(a_0^{-1}ga_0)\phi = \tilde{\Delta}^*(a_0^{-1}a)\phi$$

and

$$a\phi = aa_0\varphi = \tilde{\Delta}(aa_0)\varphi$$

which are Eqs. (2.8) and (2.9), respectively. Equations (2.6) and (2.10) demonstrate the invariance of the space spanned by the functions φ_i and ϕ_i , $i = 1, \dots, d$, under the group \mathcal{G} . From Eqs. (2.6) and (2.10) we obtain the matrices

$$D(g) = \left(\begin{array}{c|c} \Delta(g) & 0 \\ \hline 0 & \Delta^*(a_0^{-1}ga_0) \end{array} \right); \quad \forall g \in G \quad (2.11)$$

and

$$D(a) = \left(\begin{array}{c|c} 0 & \Delta(aa_0) \\ \hline \Delta^*(a_0^{-1}a) & 0 \end{array} \right); \quad a = a_0g \quad \text{or} \quad a = ga_0, \quad \forall g \in G \quad (2.12)$$

The sets of matrices in Eqs. (2.11) and (2.12) form the *corepresentation of the group* \mathcal{G} , derived from the representation Γ , with the matrices $\Delta(g)$ of the subgroup G . This corepresentation may be reducible. The corepresentation matrices obey the following set of equations [26]:

$$\begin{aligned} D(g_1)D(g_2) &= D(g_1g_2); \quad \forall g_1, g_2 \in G \\ D(g)D(a) &= D(ga); \quad \forall g \in G, \text{ and } \forall a \in a_0G \\ D(a)D^*(g) &= D(ag); \quad \forall g \in G, \text{ and } \forall a \in a_0G \\ D(a_1)D^*(a_2) &= D(a_1a_2); \quad \forall a_1, a_2 \in a_0G \end{aligned} \quad (2.13)$$

These are established by examining the action of the respective products of elements g and a on the basis functions, when the antilinear character of the elements a is taken into account. Because of the last two equalities, the mapping $\mathcal{G} \rightarrow D\Gamma$ is not a homomorphism.

The equivalence of two corepresentations.

Performing the basis transformation with a nonsingular transformation S ,

$$\tilde{S}\chi = \chi', \quad \text{with} \quad \tilde{\chi} = (\varphi, \phi) \quad (2.14)$$

where φ and ϕ are given in Eq. (2.6), we obtain

$$g\chi' = \tilde{D}'(g)\chi', \quad \text{hence} \quad D'(g) = S^{-1}D(g)S \quad (2.15)$$

and

$$a\chi' = \tilde{D}'(a)\tilde{S}\chi, \quad \text{or} \quad a\chi' = a\tilde{S}\chi = \tilde{S}^*\tilde{D}(a)\chi, \quad \text{hence} \quad D'(a) = S^{-1}D(a)S^* \quad (2.16)$$

Two corepresentations of the group \mathcal{G} , the corepresentation with the matrices $D(g)$ and $D(a)$, and the corepresentation with the matrices $D'(g)$ and $D'(a)$ are said to be equivalent if there exists a nonsingular matrix S such that

$$D'(g) = S^{-1}D(g)S; \quad \forall g \in G \quad (2.17)$$

$$D'(a) = S^{-1}D(a)S^*; \quad \forall a \in a_0G \quad (2.18)$$

The equivalence conditions in Eqs. (2.17) and (2.18) allow to replace the corepresentation matrices $D(a)$, by the matrices $\exp(i\alpha_0)D(a)$, with a real parameter α_0 , by applying the transformation $S = \exp(-i\alpha_0/2)E$, where E denotes the unit matrix. The matrices $D(g)$ remain unaltered. The matrices of the elements a then acquire the form

$$D'(a) = e^{i\alpha_0}D(a) \quad (2.19)$$

It can be shown that there is no ambiguity in the assignment of the corepresentation $D\Gamma$, derived from the representation Γ , to the group \mathcal{G} . Different choices of a_0 in the definition of \mathcal{G} lead to equivalent corepresentations [3, 26].

Reducibility of corepresentations.

If the basis χ in Eq. (2.14) can be transformed by a nonsingular transformation S so that the new basis $\chi' = \tilde{S}\chi$ is the direct sum of two subspaces which are both invariant under the group \mathcal{G} , the corep $D\Gamma$ is said to be *reducible*. If not, $D\Gamma$ is said to be *irreducible*. In the case of irreducibility, all the matrices of the corep $D'\Gamma$, equivalent to $D\Gamma$, must be in the same block-diagonal form. The two representations Γ and $\bar{\Gamma}$ may be inequivalent or equivalent. The answer to the question about the reducibility of the corep in Eqs. (2.11) and (2.12) hinges upon that.

The representations Γ and $\bar{\Gamma}$ are inequivalent.

Let us suppose that the corep matrices in Eqs. (2.11) and (2.12) are reduced by a matrix S . Since the matrices $D(g)$, $g \in G$, are in the form of the direct sum of the irreducible matrices $\Delta(g)$ and $\Delta^*(a_0^{-1}ga_0)$, their only reduced form is

$$\left(\begin{array}{c|c} X(g) & 0 \\ \hline 0 & Y(g) \end{array} \right)$$

where the matrices $X(g)$ and $Y(g)$ are equivalent to $\Delta(g)$ and $\Delta^*(a_0^{-1}ga_0)$, respectively. Writing

$$S^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

we obtain the condition in Eq. (2.17) in the form

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \left(\begin{array}{c|c} \Delta(g) & 0 \\ \hline 0 & \Delta^*(a_0^{-1}ga_0) \end{array} \right) = \begin{pmatrix} X(g) & 0 \\ \hline 0 & Y(g) \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

from which we obtain the three conditions,

$$(1) \quad p\Delta(g) = X(g)p; \quad (2) \quad s\Delta^*(a_0^{-1}ga_0) = Y(g)s$$

and

$$(3) \quad q\Delta^*(a_0^{-1}ga_0) = X(g)q = p\Delta(g)p^{-1}q$$

where in the last equality we made use of the fact that p^{-1} must exist, as it provides the equivalence transformation between $\Delta(g)$ and $X(g)$. From (1) and (3) we find that

$$(p^{-1}q)\Delta^*(a_0^{-1}ga_0) = \Delta(g)(p^{-1}q)$$

As $\Delta(g)$ and $\Delta^*(a_0^{-1}ga_0)$ were assumed to be inequivalent, by Schur's Lemma we must have $p^{-1}q = 0$, and, consequently, $q = 0$. In an analogous way we can find that $r = 0$, and then S^{-1} comes out in the form

$$S^{-1} = \begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix}$$

This matrix cannot reduce the matrices $D(a)$ in Eq. (2.12) to a block-diagonal form. Consequently, if the irreps Γ and $\bar{\Gamma}$ are inequivalent, the corep of the group \mathcal{G} derived from the irrep Γ is irreducible. We are dealing with c -type irreducible corepresentation (type 3 in [26]), with the matrices in Eqs. (2.11) and (2.12.)

The representations Γ and $\bar{\Gamma}$ are equivalent.

There exists then a nonsingular matrix N (the matrix β in [26]) such that

$$\Delta(g) = N\Delta^*(a_0^{-1}ga_0)N^{-1}, \quad \forall g \in G \quad (2.20)$$

Replacing the element g with $a_0^{-1}ga_0$ we also obtain

$$\Delta^*(a_0^{-1}ga_0) = N^*\Delta(a_0^{-2}ga_0^2)(N^{-1})^* = N^*\Delta(a_0^{-2})\Delta(g)\Delta(a_0^2)(N^{-1})^* \quad (2.21)$$

Substituting the last expression into Eq. (2.20) we obtain the equation

$$\Delta(g) = NN^* \Delta^{-1}(a_0^2) \Delta(g) \Delta(a_0^2) (N^{-1})^* N^{-1}, \quad \forall g \in G \quad (2.22)$$

Since Γ is irreducible, it follows from Schur's Lemma that $NN^* \Delta^{-1}(a_0^2) = \Lambda E$ where Λ is a constant and E is the unit matrix. Hence we obtain:

$$\Delta(a_0^2) = \Lambda^{-1} NN^*, \quad \text{and} \quad \Delta^*(a_0^2) = (\Lambda^*)^{-1} N^* N \quad (2.23)$$

In Eq. (2.20) we can put $g = a_0^2$ and we then obtain

$$\Delta(a_0^2) = N \Delta^*(a_0^2) N^{-1} \quad (2.24)$$

Substituting the right hand sides of Eqs. (2.23) into Eq. (2.24), we obtain the equalities: $\Lambda^{-1} NN^* = N(\Lambda^*)^{-1} N^* NN^{-1} = (\Lambda^*)^{-1} NN^*$, and hence $\Lambda = \Lambda^*$. Calculating the determinant of both sides of Eqs. (2.23) we obtain:

$$\Lambda = \pm \frac{|\det N \det N^*|}{|\det \Delta(a_0^2)|} = \pm 1 \quad (2.25)$$

when we assume that the irrep Γ consists of matrices with $|\det \Delta(g)| = 1$, and we remember that the matrix N can always be chosen so as to have $|\det N \det N^*| = 1$. Consequently, from Eq. (2.23) we obtain

$$NN^* = \pm \Delta(a_0^2) \quad (2.26)$$

as in the case of unitary matrices N , as in [26],[3],[11]. The reducibility of a corep depends on the sign in Eq. (2.26).

A corepresentation $D\Gamma$ is reducible if and only if the matrices $D(g)$ and $D(a)$ can simultaneously be expressed in the same block-diagonal form. The matrices $D(g)$ in Eq. (2.11) are already in a reduced form, however, it will be convenient to convert them to the form, when there are the same blocks along the diagonal. Applying the matrix

$$W = \left(\begin{array}{c|c} E & 0 \\ \hline 0 & -N^{-1} \end{array} \right) \quad (2.27)$$

with N from Eq. (2.20), and $D(a_0)$ in Eq. (2.12), we obtain:

$$D'(g) = W^{-1} D(g) W = \left(\begin{array}{c|c} \Delta(g) & 0 \\ \hline 0 & \Delta(g) \end{array} \right) \quad (2.28)$$

and

$$D'(a_0) = W^{-1} D(a_0) W^* = \left(\begin{array}{c|c} 0 & -\Delta(a_0^2)(N^{-1})^* \\ \hline -N & 0 \end{array} \right) \quad (2.29)$$

Since every element of \mathcal{G} is of the form g, a_0g or ga_0 , for $g \in G$, while $D'(a_0g) = D'(a_0)D'^*(g)$ and $D'(ga_0) = D'(g)D'(a_0)$, a nonsingular transformation V is required, which will reduce the matrices $D'(a_0)$ to block-diagonal form, leaving the matrices $D'(g)$ unaltered. That V must commute with $D'(g)$ in Eq. (2.28). Writing:

$$V^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad (2.30)$$

from the equation $V^{-1}D'(g) = D'(g)V^{-1}$ we obtain:

$$\begin{pmatrix} \alpha\Delta(g) & \beta\Delta(g) \\ \gamma\Delta(g) & \delta\Delta(g) \end{pmatrix} = \begin{pmatrix} \Delta(g)\alpha & \Delta(g)\beta \\ \Delta(g)\gamma & \Delta(g)\delta \end{pmatrix} \quad (2.31)$$

As the matrices $\Delta(g)$ are irreducible, from Schur's Lemma we find that $\alpha = \lambda E$, $\beta = \mu E$, $\gamma = \nu E$ and $\delta = \rho E$, with constant λ, μ, ν, ρ , where E is a d -dimensional unit matrix. We therefore must have

$$V^{-1} = \begin{pmatrix} \lambda E & \mu E \\ \nu E & \rho E \end{pmatrix} \quad (2.32)$$

The required existence of V implies that $\det V^{-1} \neq 0$, which leads to

$$\lambda\rho \neq \mu\nu \quad (2.33)$$

We find that

$$V = \frac{1}{2} \begin{pmatrix} E/\lambda & E/\nu \\ E/\mu & E/\rho \end{pmatrix} = E \quad (2.34)$$

where E in the matrix is a d -dimensional unit matrix, and on the right hand side E is a $2d$ -dimensional unit matrix, with

$$\lambda\rho = -\mu\nu \quad (2.35)$$

which is the condition for a reduction of a corep to be possible. With $D'(a_0)$ in Eq. (2.29), the transformed matrix $D''(a_0)$ has the form

$$D''(a_0) = V^{-1}D'(a_0)V^* = \frac{1}{2} \left(\begin{array}{c|c} -(\mu/\lambda^*)N - (\lambda/\mu^*)\Delta(a_0^2)(N^{-1})^* & -(\mu/\nu^*)N - (\lambda/\rho^*)\Delta(a_0^2)(N^{-1})^* \\ \hline -(\rho/\lambda^*)N - (\nu/\mu^*)\Delta(a_0^2)(N^{-1})^* & -(\rho/\nu^*)N - (\nu/\rho^*)\Delta(a_0^2)(N^{-1})^* \end{array} \right) \quad (2.36)$$

As the off-diagonal terms have to vanish, and from Eq. (2.35), we obtain the condition:

$$NN^* = \frac{|\lambda|^2}{|\mu|^2} \Delta(a_0^2) \quad (2.37)$$

which has the form of Eq.(2.26) with (+) sign, provided that

$$\frac{|\lambda|^2}{|\mu|^2} = 1 \quad (2.38)$$

The irreps Γ and $\bar{\Gamma}$ are equivalent and a reduction of the corepresentation in Eqs. (2.11) and (2.12) is possible.

Considering Eqs. (2.37) and (2.38) we find that a reduction of the corep in Eqs. (2.11) and (2.12) is possible when

$$NN^* = +\Delta(a_0^2) \quad (2.39)$$

Taking into account Eqs. (2.35) and (2.37), we obtain from Eq. (2.36) the reduced matrix $D''(a_0)$ in the form

$$D''(a_0) = \left(\begin{array}{c|c} (-\mu/\lambda^*)N & 0 \\ \hline 0 & (-\rho/\nu^*)N \end{array} \right) \quad (2.40)$$

Owing to Eq. (2.35), the coefficients μ/λ^* and ρ/ν^* , have the same absolute value and they can differ only by a phase factor, and hence, according to Eq. (2.18), the two blocks along the diagonal are equivalent. For a unitary N , the matrix in Eq. (2.40) turns into the customary matrix $D''(a_0)$, for example in Eq. (7.3.40) of [3], or in Eq. (1.5.40) of [11].

In order to determine the matrix connected with the element $a = ga_0$ we use the second from Eqs. (2.13) and obtain $D(a) = D(ga_0) = D(g)D(a_0)$, hence from Eqs. (2.28) and (2.40) we obtain the matrix

$$D''(ga_0) = \left(\begin{array}{c|c} -(\mu/\lambda^*)\Delta(g)N & 0 \\ \hline 0 & -(\rho/\nu^*)\Delta(g)N \end{array} \right) \quad (2.41)$$

We observe that the reduced matrices in Eqs. (2.28) and (2.41), with the two blocks in $D''(a)$ in the same form, can be obtained by applying to corep matrices in Eqs. (2.11) and (2.12) the transformation, which is analogous to that given by Kovalev and Gorbanyuk for unitary groups [20], namely:

$$V_1 = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} E & iE \\ \hline (\lambda/\mu)N^{-1} & -i(\lambda/\mu)N^{-1} \end{array} \right), \quad V_1^{-1} = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} E & (\mu/\lambda)N \\ \hline -iE & i(\mu/\lambda)N \end{array} \right) \quad (2.42)$$

Applying this transformation and taking into account the similarity transformation in Eqs. (2.17) and (2.18), we obtain corep matrices in the form:

$$D'(g) = \left(\begin{array}{c|c} \Delta(g) & 0 \\ \hline 0 & \Delta(g) \end{array} \right), \quad D'(ga_0) = e^{i\alpha_0} \left(\begin{array}{c|c} (\mu/\lambda)\Delta(g)N & 0 \\ \hline 0 & (\mu/\lambda)\Delta(g)N \end{array} \right) \quad (2.43)$$

and

$$D'(a_0g) = e^{i\alpha_0} \left(\begin{array}{c|c} (\mu/\lambda)N\Delta^*(g) & 0 \\ \hline 0 & (\mu/\lambda)N\Delta^*(g) \end{array} \right) \quad (2.44)$$

With $g = E$, we obtain

$$D'(a_0) = e^{i\alpha_0} \left(\begin{array}{c|c} (\mu/\lambda)N & 0 \\ \hline 0 & (\mu/\lambda)N \end{array} \right) \quad (2.45)$$

which replaces Eq. (2.40), in which the two block matrices appear with opposite signs. When the matrix N is unitary, and we put $\mu/\lambda = 1$, the transformation V_1 in Eq. (2.42) turns into Eq. (8.11a) in [20], or into Eq. (1.5.43) in [11]. In general we have $\mu/\lambda = \exp(i\xi)$, with a real ξ , and this exponential factor can be absorbed by the factor $\exp(i\alpha_0)$.

Renaming the functions ϕ_i in Eq. (2.4) of the original corepresentation,

$$\phi_i = a_0\varphi_i = \varphi_{d+i}, \quad i = 1, \dots, d \quad (2.46)$$

and utilizing the transformation V_1 in Eq. (2.42), we obtain the basis functions of the two blocks, with labels (1) and (2),

$$\psi_j^{(1)} = \frac{1}{\sqrt{2}} \left(\varphi_j + \frac{\lambda}{\mu} \sum_{i=1}^d (\tilde{N}^{-1})_{ij} \varphi_{d+i} \right), \quad \psi_j^{(2)} = \frac{i}{\sqrt{2}} \left(\varphi_j - \frac{\lambda}{\mu} \sum_{i=1}^d (\tilde{N}^{-1})_{ij} \varphi_{d+i} \right) \quad (2.47)$$

In the case of unitary groups G , when the original basis functions φ_j in Eq. (2.2) are orthogonal, the basis functions $\psi_j^{(1)}$, $j = 1, \dots, d$, also are orthogonal, and the same holds for the functions $\psi_j^{(2)}$. These two sets of functions need not be mutually orthogonal, however.

The irreps Γ and $\bar{\Gamma}$ are equivalent, however a reduction of the corepresentation in Eqs. (2.11) and (2.12) is impossible.

According to Eq. (2.26) with the $(-)$ sign, we now have:

$$NN^* = -\Delta(a_0^2) \quad (2.48)$$

and from Eq. (2.29) we obtain

$$D'(a_0) = \left(\begin{array}{c|c} 0 & N \\ \hline -N & 0 \end{array} \right) \quad (2.49)$$

With $a = ga_0$, hence $g = aa_0^{-1}$, and $D'(a) = D'(g)D'(a_0)$, with $D'(g)$ in Eq. (2.28) and $D'(a_0)$ in Eq. (2.49), we obtain

$$D'(ga_0) = \left(\begin{array}{c|c} 0 & \Delta(g)N \\ \hline -\Delta(g)N & 0 \end{array} \right) \quad (2.50)$$

With $a = a_0g$, hence $g = a_0^{-1}a$, from the third of Eqs. (2.13) and from Eq. (2.49) we obtain the matrix

$$D'(a_0g) = D'(a_0)D'^*(g) = \left(\begin{array}{c|c} 0 & N\Delta^*(g) \\ \hline -N\Delta^*(g) & 0 \end{array} \right) \quad (2.51)$$

When the similarity transformation in Eq. (2.18) is applied to the above matrices connected with the coset a_0G , they acquire the factor $\exp(i\alpha_0)$.

We observe that Eqs. (2.28), (2.50) and (2.51) can be obtained by applying to the corep matrices in Eqs. (2.11) and (2.12) the transformation:

$$V_2 = \left(\begin{array}{c|c} iE & 0 \\ \hline 0 & iN^{-1} \end{array} \right) \quad (2.52)$$

When the matrix N is unitary, V_2 turns into the transformation given by Kovalev and Gorbanyuk in Eq. (8.10) of [20], (or Eq. (1.5.50) of [11]).

The basis functions transforming according to the matrices in Eqs. (2.28), (2.50) and (2.51) are determined from the equality,

$$\tilde{V}_2 \begin{pmatrix} \varphi \\ \phi \end{pmatrix} = \begin{pmatrix} i\varphi \\ i\tilde{N}^{-1}\phi \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{2d} \end{pmatrix} \quad (2.53)$$

with ϕ_i in (2.46) or,

$$\begin{aligned} \psi_j &= i\varphi_j, \quad j = 1, \dots, d \\ \psi_{d+j} &= -i \sum_{k=1}^d (\tilde{N}^{-1})_{jk} \varphi_{d+k}, \quad j = 1, \dots, d, \quad \text{with} \quad \varphi_{d+k} = a_0 \varphi_k \end{aligned} \quad (2.54)$$

The corepresentation formulas hold for single-valued as well as for double-valued representations Γ of the subgroup G .

3 Conclusions

We have presented the corepresentation theory without the assumption of the unitarity of the subgroup G of the group $G + a_0G$, where a_0 denotes an antilinear operation. The formulas of the corepresentation theory with unitary groups G can be obtained from this presentation.

Acknowledgments

We are indebted to Professor Zbigniew Oziewicz from the Universidad Nacional Autónoma de México for the discussions concerning the mappings with antilinear operations.

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